

HODGE POLYNOMIALS OF THE MODULI SPACES OF RANK 3 PAIRS

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A holomorphic triple (E_1, E_2, ϕ) on X consists of two holomorphic vector bundles E_1 and E_2 over X and a holomorphic map $\phi: E_2 \rightarrow E_1$. There is a concept of stability for triples which depends on a real parameter σ . In this paper, we determine the Hodge polynomials of the moduli spaces of σ -stable triples with $\text{rk}(E_1) = 3$, $\text{rk}(E_2) = 1$, using the theory of mixed Hodge structures. This gives in particular the Poincaré polynomials of these moduli spaces. As a byproduct, we recover the Hodge polynomial of the moduli space of odd degree rank 3 stable vector bundles.

1. INTRODUCTION

Let X be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. A holomorphic triple $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X and a holomorphic map $\phi: E_2 \rightarrow E_1$. There is a concept of stability for a triple which depends on the choice of a parameter $\sigma \in \mathbb{R}$. This gives a collection of moduli spaces \mathcal{N}_σ , which have been studied in [1, 2, 10, 11, 12]. The range of the parameter σ is an interval $I \subset \mathbb{R}$ split by a finite number of *critical values* σ_c in such a way that, when σ moves without crossing a critical value, then \mathcal{N}_σ remains unchanged, but when σ crosses a critical value, \mathcal{N}_σ undergoes a transformation which we call *flip*. The study of this process allows to obtain geometrical information on all the moduli spaces \mathcal{N}_σ .

For a projective smooth variety Z , the Hodge polynomial is defined as

$$e(Z)(u, v) = \sum_{p, q} h^{p, q}(Z) u^p v^q,$$

where $h^{p, q}(Z) = \dim H^{p, q}(Z)$ are the Hodge numbers of Z . In particular, the Poincaré polynomial equals $P_t(Z) = e(Z)(t, t)$. The theory of mixed Hodge structures introduced by Deligne [5] allows to extend the definition of Hodge polynomials to any algebraic variety (non-smooth or non-complete). Using this, in [11, 12] the Hodge polynomials of the moduli spaces of triples when the ranks of E_1 and E_2 are at most 2 were found.

When the rank of E_2 is one, we have the so-called *pairs*, studied in [14, 9, 11]. The moduli spaces of pairs are smooth projective varieties for non-critical values of σ . In this paper, we study the moduli spaces of pairs of rank 3 (i.e. when the rank of E_1 is 3), computing their Hodge polynomials. This gives, in particular, the Poincaré polynomials of these moduli spaces. Our main result is

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Theorem 1.1. *Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(1, 3, d_1, d_2)$ be the moduli space of σ -stable triples of type $(1, 3, d_1, d_2)$. Assume that $\sigma \in I$ is non-critical. Then the Hodge polynomial is*

$$\begin{aligned} e(\mathcal{N}_\sigma) = & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} \\ & \cdot \left[\left(\frac{(uv)^{2d_1-2d_2-2n_0}x^{n_0}}{1-(uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0}x^{n_0}}{1-(uv)^3x} \right) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right. \\ & + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1}x^{\bar{n}_0}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} \right. \\ & \left. \left. + \frac{(uv)^{2g-2-2d_1+3\bar{n}_0}x^{\bar{n}_0}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1-d_2+\bar{n}_0/2}x^{\bar{n}_0}}{(1-(uv)^2x)(1-(uv)^{-1}x)} \right) \right], \end{aligned}$$

where $n_0 = \lceil \frac{\sigma+d_1+d_2}{2} \rceil$ and $\bar{n}_0 = 2\lfloor \frac{n_0+1}{2} \rfloor$ (where $[x]$ denotes the integer part of $x \in \mathbb{R}$).

In particular, the Poincaré polynomial is

$$\begin{aligned} P_t(\mathcal{N}_\sigma) = & (1+t)^{4g} \operatorname{coeff}_{x^0} \frac{(1+tx)^{2g}}{(1-x)(1-t^2x)x^{d_1-d_2}} \\ & \cdot \left[\left(\frac{t^{4d_1-4d_2-4n_0}x^{n_0}}{1-t^{-4}x} - \frac{t^{4g-4-4d_1+6n_0}x^{n_0}}{1-t^6x} \right) \cdot \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)^2(1-t^4)} \right. \\ & \left. + \frac{t^{2g-2}(1+t)^{2g}}{(1-t^2)^2(1+t^2)} \left(\frac{t^{4d_1-4d_2-4\bar{n}_0+2}x^{\bar{n}_0}}{(1-t^{-4}x)(1-t^{-2}x)} + \frac{t^{4g-4-4d_1+6\bar{n}_0}x^{\bar{n}_0}}{(1-t^6x)(1-t^4x)} - \frac{(1+t^2)t^{2g-2-2d_2+\bar{n}_0}x^{\bar{n}_0}}{(1-t^4x)(1-t^{-2}x)} \right) \right]. \end{aligned}$$

Let $M(n, d)$ denote the moduli space of polystable vector bundles of rank n and degree d over X . This moduli space is projective. We also denote by $M^s(n, d)$ the open subset of stable bundles, which is smooth of dimension $n^2(g-1)+1$. If $\gcd(n, d) = 1$, then $M(n, d) = M^s(n, d)$.

For the smallest possible values of the parameter $\sigma \in I$, there is a map from $\mathcal{N}_\sigma(1, 3, d_1, d_2)$ to $M(3, d)$ given by $(E_1, E_2, \phi) \mapsto E_1$. The study of this map allows us to recover the Hodge polynomial of $M(3, d)$ when $d \not\equiv 0 \pmod{3}$. This was found previously in [8] by other methods.

Theorem 1.2. *Assume that $d \not\equiv 0 \pmod{3}$. Then the Hodge polynomial of $M(3, d)$ is*

$$\begin{aligned} e(M(3, d)) = & \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)^2(1-(uv)^3)} \left((1+u)^g(1+v)^g(1+uv)^2(uv)^{2g-1}(1+u^2v)^g(1+uv^2)^g \right. \\ & \left. - (1+u)^{2g}(1+v)^{2g}(uv)^{3g-1}(1+uv+u^2v^2) + (1+u^2v^3)^g(1+u^3v^2)^g(1+u^2v)^g(1+uv^2)^g \right). \end{aligned}$$

In particular, the Poincaré polynomial is

$$P_t(M(3, d)) = (1+t)^{4g} \frac{(1+t)^{2g}(1+t^2)^2 t^{4g-2}(1+t^3)^{2g} - (1+t)^{4g} t^{6g-2}(1+t^2+t^4) + (1+t^5)^{2g}(1+t^3)^{2g}}{(1-t^2)(1-t^4)^2(1-t^6)}.$$

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2. HODGE POLYNOMIALS

Let us start by recalling the Hodge-Deligne theory of algebraic varieties over \mathbb{C} . Let H be a finite-dimensional complex vector space. A *pure Hodge structure of weight k* on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, the bar denoting complex conjugation in H . We denote

$$h^{p,q}(H) = \dim H^{p,q},$$

which is called the Hodge number of type (p, q) . A Hodge structure of weight k on H gives rise to the so-called *Hodge filtration* F on H , where

$$F^p = \bigoplus_{s \geq p} H^{s, p-s},$$

which is a descending filtration. Note that $\mathrm{Gr}_F^p H = F^p / F^{p+1} = H^{p,q}$.

Let H be a finite-dimensional complex vector space. A *(mixed) Hodge structure* over H consists of an ascending weight filtration W on H and a descending Hodge filtration F on H such that F induces a pure Hodge filtration of weight k on each $\mathrm{Gr}_k^W H = W_k / W_{k-1}$. Again we define

$$h^{p,q}(H) = \dim H^{p,q}, \quad \text{where } H^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H.$$

Deligne has shown [5] that, for each complex algebraic variety Z , the cohomology $H^k(Z)$ and the cohomology with compact support $H_c^k(Z)$ both carry natural Hodge structures. If Z is a compact smooth projective variety (hence compact Kähler) then the Hodge structure $H^k(Z)$ is pure of weight k and coincides with the classical Hodge structure given by the Hodge decomposition of harmonic forms into (p, q) types.

Definition 2.1. For *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible), we define the Hodge numbers as

$$h_c^{k,p,q}(Z) = h^{p,q}(H_c^k(Z)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Z).$$

Introduce the Euler characteristic

$$\chi_c^{p,q}(Z) = \sum (-1)^k h_c^{k,p,q}(Z)$$

The *Hodge polynomial* of Z is defined [7] as

$$e(Z) = e(Z)(u, v) = \sum_{p,q} (-1)^{p+q} \chi_c^{p,q}(Z) u^p v^q.$$

If Z is smooth and projective then the mixed Hodge structure on $H^k(Z)$ is pure of weight k , so $\mathrm{Gr}_k^W H_c^k(Z) = H_c^k(Z) = H^k(Z)$ and the other pieces $\mathrm{Gr}_m^W H_c^k(Z) = 0$, $m \neq k$. So

$$\chi_c^{p,q}(Z) = (-1)^{p+q} h^{p,q}(Z),$$

where $h^{p,q}(Z)$ is the usual Hodge number of Z . In this case,

$$e(Z)(u, v) = \sum_{p,q} h^{p,q}(Z) u^p v^q$$

is the (usual) Hodge polynomial of Z . Note that in this case, the Poincaré polynomial of Z is

$$P_Z(t) = \sum_k b^k(Z) t^k = \sum_k \left(\sum_{p+q=k} h^{p,q}(Z) \right) t^k = e(Z)(t, t). \quad (2.1)$$

where $b^k(Z)$ is the k -th Betti number of Z .

Theorem 2.2 ([11, Theorem 2.2][6]). *Let Z be a complex algebraic variety. Suppose that Z is a finite disjoint union $Z = Z_1 \cup \dots \cup Z_n$, where the Z_i are algebraic subvarieties. Then*

$$e(Z) = \sum_i e(Z_i).$$

□

The following Hodge polynomials will be needed later:

(1) For the projective space \mathbb{P}^{n-1} , we have

$$e(\mathbb{P}^{n-1}) = 1 + uv + (uv)^2 + \dots + (uv)^{n-1} = \frac{1 - (uv)^n}{1 - uv}.$$

- (2) Let $\text{Jac}^d X$ be the Jacobian of (any) degree d of a (smooth, projective) complex curve X of genus g . Then

$$e(\text{Jac}^d X) = (1+u)^g(1+v)^g.$$

- (3) Let X be a curve (smooth, projective) complex curve of genus g , and $k \geq 1$. The Hodge polynomial of the symmetric product $\text{Sym}^k X$ is computed in [3],

$$e(\text{Sym}^k X) = \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^k}.$$

- (4) The Hodge polynomial of the Grassmannian $\text{Gr}(k, N)$ is given by [12, Lemma 2.5],

$$e(\text{Gr}(k, N)) = \frac{(1-(uv)^{N-k+1}) \cdots (1-(uv)^{N-1})(1-(uv)^N)}{(1-uv) \cdots (1-(uv)^{k-1})(1-(uv)^k)}.$$

- (5) Suppose that $\pi : Z \rightarrow Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then [11, Lemma 2.3]

$$e(Z) = e(F) e(Y).$$

In particular this is true for $Z = F \times Y$.

- (6) Suppose that $\pi : Z \rightarrow Y$ is a map between quasi-projective varieties which is a locally trivial fiber bundle in the usual topology, with fibers being projective spaces $F = \mathbb{P}^N$ for some $N > 0$. Then [12, Lemma 2.4]

$$e(Z) = e(F) e(Y).$$

- (7) Let M be a smooth projective variety. Consider the algebraic variety $Z = (M \times M)/\mathbb{Z}_2$, where \mathbb{Z}_2 acts as $(x, y) \mapsto (y, x)$. The Hodge polynomial of Z is [12, Lemma 2.6]

$$e(Z) = \frac{1}{2} \left(e(M)(u, v)^2 + e(M)(-u^2, -v^2) \right).$$

3. MODULI SPACES OF TRIPLES

3.1. Holomorphic triples. Let X be a smooth projective curve of genus $g \geq 2$ over \mathbb{C} . A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X , of ranks n_1 and n_2 and degrees d_1 and d_2 , respectively, and a holomorphic map $\phi : E_2 \rightarrow E_1$. We refer to (n_1, n_2, d_1, d_2) as the *type* of T , to (n_1, n_2) as the *rank* of T , and to (d_1, d_2) as the *degree* of T .

A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1, \end{array}$$

where the vertical arrows are holomorphic maps. A triple $T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if $E'_1 \subset E_1$ and $E'_2 \subset E_2$ are subbundles, $\phi(E'_2) \subset E'_1$ and $\phi' = \phi|_{E'_2}$. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$. The quotient triple $T'' = T/T'$ is given by $E''_1 = E_1/E'_1$, $E''_2 = E_2/E'_2$ and $\phi'' : E''_2 \rightarrow E''_1$ being the map induced by ϕ . We usually denote by (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, the types of the subtriple T' and the quotient triple T'' .

Definition 3.1. For any $\sigma \in \mathbb{R}$ the σ -slope of T is defined by

$$\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \sigma \frac{n_2}{n_1 + n_2}.$$

To shorten the notation, we define the μ -slope and λ -slope of the triple T as $\mu = \mu(E_1 \oplus E_2) = \frac{d_1 + d_2}{n_1 + n_2}$ and $\lambda = \frac{n_2}{n_1 + n_2}$, so that $\mu_\sigma(T) = \mu + \sigma\lambda$.

Definition 3.2. We say that a triple $T = (E_1, E_2, \phi)$ is σ -stable if

$$\mu_\sigma(T') < \mu_\sigma(T),$$

for any proper subtriple $T' = (E'_1, E'_2, \phi')$. We define σ -semistability by replacing the above strict inequality with a weak inequality. A triple is called σ -polystable if it is the direct sum of σ -stable

triples of the same σ -slope. It is σ -unstable if it is not σ -semistable, and *strictly* σ -semistable if it is σ -semistable but not σ -stable. A σ -destabilizing subtriple $T' \subset T$ is a proper subtriple satisfying $\mu_\sigma(T') \geq \mu_\sigma(T)$.

We denote by

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$$

the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) , and drop the type from the notation when it is clear from the context. The open subset of σ -stable triples is denoted by $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$. This moduli space is constructed in [1] by using dimensional reduction. A direct construction is given by Schmitt [13] using geometric invariant theory.

There are certain necessary conditions in order for σ -semistable triples to exist. Let $\mu_i = \mu(E_i) = d_i/n_i$ stand for the slope of E_i , for $i = 1, 2$. We write

$$\begin{aligned} \sigma_m &= \mu_1 - \mu_2, \\ \sigma_M &= \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right) (\mu_1 - \mu_2), \quad \text{if } n_1 \neq n_2. \end{aligned}$$

Proposition 3.3. [2] *The moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is a complex projective variety. Let I denote the interval $I = [\sigma_m, \sigma_M]$ if $n_1 \neq n_2$, or $I = [\sigma_m, \infty)$ if $n_1 = n_2$. A necessary condition for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is that $\sigma \in I$. \square*

The moduli space \mathcal{N}_σ for the smallest possible values of the parameter can be explicitly described. We refer to the value of σ given by $\sigma = \sigma_m^+ = \sigma_m + \epsilon$ ($\epsilon > 0$ small) as *small*.

Proposition 3.4 ([11, Proposition 4.10]). *There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \rightarrow M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \phi)$ to (E_1, E_2) . If $\gcd(n_1, d_1) = 1$, $\gcd(n_2, d_2) = 1$ and $\mu_1 - \mu_2 > 2g - 2$, then $\mathcal{N}_{\sigma_m^+}$ is a projective bundle over $M(n_1, d_1) \times M(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g - 1) - 1$. \square

To study the dependence of the moduli spaces \mathcal{N}_σ on the parameter, we need to introduce the concept of critical value [1, 11].

Definition 3.5. The values of $\sigma_c \in I$ for which there exist $0 \leq n'_1 \leq n_1$, $0 \leq n'_2 \leq n_2$, d'_1 and d'_2 , with $n'_1 n_2 \neq n_1 n'_2$, such that

$$\sigma_c = \frac{(n_1 + n_2)(d'_1 + d'_2) - (n'_1 + n'_2)(d_1 + d_2)}{n'_1 n_2 - n_1 n'_2}, \quad (3.1)$$

are called *critical values*.

Given a triple $T = (E_1, E_2, \phi)$, the condition of σ -(semi)stability for T can only change when σ crosses a critical value. If $\sigma = \sigma_c$ as in (3.1) and if T has a subtriple $T' \subset T$ of type (n'_1, n'_2, d'_1, d'_2) , then $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$ and

- (1) if $\lambda' > \lambda$ (where λ' is the λ -slope of T'), then T is not σ -stable for $\sigma > \sigma_c$,
- (2) if $\lambda' < \lambda$, then T is not σ -stable for $\sigma < \sigma_c$.

Note that $n'_1 n_2 \neq n_1 n'_2$ is equivalent to $\lambda' \neq \lambda$.

Proposition 3.6 ([2, Proposition 2.6]). *Fix (n_1, n_2, d_1, d_2) . Then*

- (1) *The critical values are a finite number of values $\sigma_c \in I$.*
- (2) *The stability and semistability criteria for two values of σ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*
- (3) *If σ is not a critical value and $\gcd(n_1, n_2, d_1 + d_2) = 1$, then σ -semistability is equivalent to σ -stability, i.e., $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$.*

\square

3.2. Extensions and deformations of triples. The homological algebra of triples is controlled by the hypercohomology of a certain complex of sheaves which appears when studying infinitesimal deformations [2, Section 3]. Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, respectively. Let $\text{Hom}(T'', T')$ denote the linear space of homomorphisms from T'' to T' , and let $\text{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 \longrightarrow 0 \\ & & \phi' \uparrow & & \phi \uparrow & & \phi'' \uparrow \\ 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 \longrightarrow 0. \end{array}$$

To analyze $\text{Ext}^1(T'', T')$ one considers the complex of sheaves

$$C^\bullet(T'', T') : (E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2) \xrightarrow{c} E''_2^* \otimes E'_1, \quad (3.2)$$

where the map c is defined by

$$c(\psi_1, \psi_2) = \phi' \psi_2 - \psi_1 \phi''.$$

We introduce the following notation:

$$\begin{aligned} \mathbb{H}^i(T'', T') &= \mathbb{H}^i(C^\bullet(T'', T')), \\ h^i(T'', T') &= \dim \mathbb{H}^i(T'', T'), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned}$$

Proposition 3.7 ([2, Proposition 3.1]). *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(T'', T') &\cong \mathbb{H}^0(T'', T'), \\ \text{Ext}^1(T'', T') &\cong \mathbb{H}^1(T'', T'), \end{aligned}$$

and a long exact sequence associated to the complex $C^\bullet(T'', T')$:

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(T'', T') \longrightarrow H^0((E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2)) \longrightarrow H^0(E''_2^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^1(T'', T') \longrightarrow H^1((E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2)) \longrightarrow H^1(E''_2^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^2(T'', T') \longrightarrow 0. \end{aligned}$$

□

Proposition 3.8 ([2, Proposition 3.2]). *For any holomorphic triples T' and T'' we have*

$$\begin{aligned} \chi(T'', T') &= \chi(E''_1^* \otimes E'_1) + \chi(E''_2^* \otimes E'_2) - \chi(E''_2^* \otimes E'_1) \\ &= (1 - g)(n''_1 n'_1 + n''_2 n'_2 - n''_2 n'_1) + n''_1 d'_1 - n'_1 d''_1 + n''_2 d'_2 - n'_2 d''_2 - n''_2 d'_1 + n'_1 d''_2, \end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ is the Euler characteristic of E . □

Lemma 3.9 ([2, Proposition 3.5]). *Suppose that T' and T'' are σ -semistable, for some value of σ .*

- (1) *If $\mu_\sigma(T') < \mu_\sigma(T'')$ then $\mathbb{H}^0(T'', T') = 0$.*
- (2) *If $\mu_\sigma(T') = \mu_\sigma(T'')$ and T', T'' are σ -stable, then*

$$\mathbb{H}^0(T'', T') \cong \begin{cases} \mathbb{C} & \text{if } T' \cong T'' \\ 0 & \text{if } T' \not\cong T'' \end{cases}.$$

□

Lemma 3.10. *If $T'' = (E''_1, E''_2, \phi'')$ is an injective triple, that is $\phi'' : E''_2 \rightarrow E''_1$ is injective, then $\mathbb{H}^2(T'', T') = 0$.*

Proof. Since $E''_2 \rightarrow E''_1$ is injective, we have that $E''_2 \otimes E''_2^* \otimes K \rightarrow E''_1 \otimes E''_2^* \otimes K$ is injective as well. Therefore $H^0(E''_2 \otimes E''_2^* \otimes K) \rightarrow H^0(E''_1 \otimes E''_2^* \otimes K)$ is a monomorphism. Taking duals, $H^1(E''_2^* \otimes E'_2) \rightarrow H^1(E''_2^* \otimes E'_1)$ is an epimorphism. Proposition 3.7 implies that $\mathbb{H}^2(T'', T') = 0$. □

Since the space of infinitesimal deformations of a triple T is isomorphic to $\mathbb{H}^1(T, T)$, the previous results also apply to studying deformations of a holomorphic triple T .

Theorem 3.11 ([2, Theorem 3.8]). *Let $T = (E_1, E_2, \phi)$ be an σ -stable triple of type (n_1, n_2, d_1, d_2) .*

- (1) *The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(T, T)$.*
- (2) *If $\mathbb{H}^2(T, T) = 0$, then the moduli space of σ -stable triples is smooth in a neighbourhood of the point defined by T .*
- (3) *At a smooth point $T \in \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$ the dimension of the moduli space of σ -stable triples is*

$$\begin{aligned} \dim \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2) &= h^1(T, T) = 1 - \chi(T, T) \\ &= (g-1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1. \end{aligned}$$

- (4) *Let $T = (E_1, E_2, \phi)$ be a σ -stable triple. If T is an injective triple, then the moduli space is smooth at T .*

□

4. DESCRIPTION OF THE FLIP LOCI

Fix the type (n_1, n_2, d_1, d_2) for the moduli spaces of holomorphic triples. We want to describe the differences between two spaces $\mathcal{N}_{\sigma_1}^s$ and $\mathcal{N}_{\sigma_2}^s$ when σ_1 and σ_2 are separated by a critical value. Let $\sigma_c \in I$ be a critical value and set

$$\sigma_c^+ = \sigma_c + \epsilon, \quad \sigma_c^- = \sigma_c - \epsilon,$$

where $\epsilon > 0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-, σ_c^+) .

Definition 4.1. We define the *flip loci* as

$$\begin{aligned} \mathcal{S}_{\sigma_c^+} &= \{T \in \mathcal{N}_{\sigma_c^+} \mid T \text{ is } \sigma_c^- \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^+}, \\ \mathcal{S}_{\sigma_c^-} &= \{T \in \mathcal{N}_{\sigma_c^-} \mid T \text{ is } \sigma_c^+ \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^-}. \end{aligned}$$

Note that $\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}$.

For $\sigma_c = \sigma_m$, $\mathcal{N}_{\sigma_m^-}$ is empty, hence $\mathcal{N}_{\sigma_m^+} = \mathcal{S}_{\sigma_m^+}$. Analogously, when $n_1 \neq n_2$, $\mathcal{N}_{\sigma_M^+}$ is empty and $\mathcal{N}_{\sigma_M^-} = \mathcal{S}_{\sigma_M^-}$.

We shall describe geometrically the flip loci $\mathcal{S}_{\sigma_c^\pm}$, for a critical value σ_c , in the situations which suffice for our purposes. In order to do this, recall that for a properly σ_c -semistable triple T , there is a Jordan-Hölder filtration

$$0 \subset T_1 \subset T_2 \subset \cdots \subset T_r = T, \quad (4.1)$$

where $\bar{T}_i = T_i/T_{i-1}$, $i = 1, \dots, r$, are σ_c -stable triples. Although the filtration (4.1) is not uniquely defined, the graded triple associated to T ,

$$\text{gr}_{\sigma_c}(T) = \bigoplus_{i=1}^r \bar{T}_i \quad (4.2)$$

is well-defined, up to order of the summands. To describe $\mathcal{S}_{\sigma_c^\pm}$, we shall stratify them according to the different possibilities for (4.2). We do this in the cases $r \leq 3$.

The case where $r = 2$ is specially simple.

Proposition 4.2. *Let σ_c be a critical value given by (n'_1, n'_2, d'_1, d'_2) in (3.1), and let $(n''_1, n''_2, d''_1, d''_2) = (n_1 - n'_1, n_2 - n'_2, d_1 - d'_1, d_2 - d'_2)$. Let $X^+ \subset \mathcal{S}_{\sigma_c^+}$ (resp. $X^- \subset \mathcal{S}_{\sigma_c^-}$) be the subset of those triples T such that T sits in a non-split exact sequence*

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \quad (4.3)$$

where $T' \in \mathcal{N}'_{\sigma_c} = \mathcal{N}_\sigma(n'_1, n'_2, d'_1, d'_2)$ and $T'' \in \mathcal{N}''_{\sigma_c} = \mathcal{N}_\sigma(n''_1, n''_2, d''_1, d''_2)$ are both σ_c -stable, and $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Assume that $\mathbb{H}^2(T'', T') = 0$, for every $T' \in \mathcal{N}'_{\sigma_c, s}$, $T'' \in \mathcal{N}''_{\sigma_c, s}$. Then X^+ (resp. X^-) is the projectivization of a bundle of rank $-\chi(T'', T')$ over $\mathcal{N}'_{\sigma_c, s} \times \mathcal{N}''_{\sigma_c, s}$.

Proof. Let us do the case of X^+ , the other one being analogous. First of all, note that for any $T \in X^+$, there is a unique exact sequence like (4.3) in which T sits. For consider any proper non-trivial $\tilde{T} \subset T$ with the same σ_c -slope as T , compose with the projection $T \rightarrow T''$ to get a map $\tilde{T} \rightarrow T''$ between triples of the same σ_c -slope. As T'' is σ_c -stable, either this map is zero or an epimorphism. In the first case, $\tilde{T} \subset T'$ and both are non-zero triples of the same σ_c -slope, so $\tilde{T} = T'$ by σ_c -stability of T' . In the second case, we have a short exact sequence $0 \rightarrow \tilde{T}' \rightarrow \tilde{T} \rightarrow T'' \rightarrow 0$, where $\tilde{T}' \subset T'$ has the same σ_c -slope as T', T'' and \tilde{T} . Therefore $\tilde{T}' = 0$ (since \tilde{T} is properly contained in T), and hence $\tilde{T} \cong T''$. This gives a splitting of the exact sequence (4.3), contrary to our assumption.

The above implies that X^+ is parametrized by the extensions (4.3). Now as $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$ are both σ_c -stable and non-isomorphic (for instance, because $\lambda' \neq \lambda''$), Lemma 3.9 implies that $\mathbb{H}^0(T'', T') = 0$. Then $\mathbb{H}^1(T'', T')$ has constant dimension equal to

$$\dim \mathbb{H}^1(T'', T') = -\chi(T'', T').$$

Therefore the extensions give a vector bundle of rank $-\chi(T'', T')$ over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$.

If two extensions give rise to the same triple T , then the uniqueness of the subtriple T' yields the existence of a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' \longrightarrow 0 \\ & & \downarrow & & \cong \downarrow & & \downarrow \\ 0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' \longrightarrow 0, \end{array}$$

where the left and right vertical arrows are automorphisms of T' and T'' respectively. Since both of them are σ_c -stable, we have that $\text{Aut}(T') = \text{Aut}(T'') = \mathbb{C}^*$. The action of $\text{Aut}(T') \times \text{Aut}(T'')$ on $\text{Ext}^1(T'', T') - \{0\}$ factors through the action of \mathbb{C}^* by multiplication on the fibers $\mathbb{H}^1(T'', T') - \{0\}$. So $X^+ \rightarrow \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ is a projective bundle with fibers $\mathbb{P}\mathbb{H}^1(T'', T')$. \square

To study the flips in the case of triples of rank $(3, 1)$, we shall need to deal also with σ_c -semistable triples T such that (4.2) has $r = 3$ terms. From now on, assume that $\gcd(n_1, n_2, d_1 + d_2) = 1$, so that $\mathcal{N}_{\sigma}^s = \mathcal{N}_{\sigma}$ for non-critical values σ .

On the following, fix a critical value σ_c given by (n'_1, n'_2, d'_1, d'_2) in (3.1), and let $(n''_1, n''_2, d''_1, d''_2) = (n_1 - n'_1, n_2 - n'_2, d_1 - d'_1, d_2 - d'_2)$. Suppose that $T \in \mathcal{S}_{\sigma_c}^+$ (resp. $T \in \mathcal{S}_{\sigma_c}^-$) is a properly σ_c -semistable triple with $r = 3$ terms in the Jordan-Hölder filtration. Then T sits in a non-split exact sequence

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \quad (4.4)$$

where T' is σ_c -semistable, T'' is σ_c -stable and $\lambda' < \lambda$ (resp. T' is σ_c -stable, T'' is σ_c -semistable and $\lambda' > \lambda$). For this it is enough to take $T' = T_{r-1}$ (resp. $T = T_1$) in the Jordan-Hölder filtration (4.1).

Moreover, there is an exact sequence

$$0 \rightarrow T_1 \rightarrow T' \rightarrow T_2 \rightarrow 0 \quad (\text{resp. } 0 \rightarrow T_1 \rightarrow T'' \rightarrow T_2 \rightarrow 0) \quad (4.5)$$

where T_1, T_2 are σ_c -stable triples of the same σ_c -slope. The sequence (4.5) may be split or non-split. Note that the graded triple associated to T is $T'' \oplus T_1 \oplus T_2$ (resp. $T' \oplus T_1 \oplus T_2$).

Finally, we shall denote by $\mathcal{N}_{\sigma_c}^1$ and $\mathcal{N}_{\sigma_c}^2$ the moduli spaces of triples of the types determined by T_1, T_2 , respectively. Let λ_1, λ_2 denote their λ -slopes. Notice that $\lambda_1 < \lambda$ (resp. $\lambda_2 < \lambda$).

We stratify $\mathcal{S}_{\sigma_c}^{\pm}$ according to whether (4.5) is split or non-split, and to whether $T_1 \cong T_2$ or $T_1 \not\cong T_2$.

Proposition 4.3. *With the situation above, let $X^+ \subset \mathcal{S}_{\sigma_c}^+$ (resp. $X^- \subset \mathcal{S}_{\sigma_c}^-$) be the subset of those triples T which sit in a non-split exact sequence (4.4), where the exact sequence (4.5) is non-split and $T_1 \not\cong T_2$.*

Let $U \subset \mathcal{N}_{\sigma_c}^{1,s} \times \mathcal{N}_{\sigma_c}^{2,s}$ be the open set consisting of those (T_1, T_2) with $T_1 \not\cong T_2$ (which is the whole space in case the types are different). Assume that $\lambda_2 < \lambda$ and $\mathbb{H}^2(T'', T_1) = \mathbb{H}^2(T'', T_1) = \mathbb{H}^2(T_2, T_1) = 0$, for every $T'' \in \mathcal{N}_{\sigma_c}^{''s}$, $(T_1, T_2) \in U$ (resp. $\lambda_1 < \lambda$ and $\mathbb{H}^2(T_1, T') = \mathbb{H}^2(T_2, T') = \mathbb{H}^2(T_2, T_1) = 0$, for every $T' \in \mathcal{N}_{\sigma_c}^{',s}$, $(T_1, T_2) \in Y$). Then

- (1) *The space Y parametrizing the triples T' (resp. T'') is the projectivization of a bundle of rank $-\chi(T_2, T_1)$ over U .*

- (2) X^+ (resp. X^-) is a bundle over $Y \times \mathcal{N}_{\sigma_c}^{\prime, s}$ (resp. $\mathcal{N}_{\sigma_c}^{\prime, s} \times Y$) with fibers $\mathbb{P}^{a-1} - \mathbb{P}^{b-1}$, $a = -\chi(T'', T')$, $b = -\chi(T'', T_1)$ (resp. $a = -\chi(T'', T')$, $b = -\chi(T_2, T')$).

Proof. We shall do the case of X^- , the other being analogous. By assumption $T_1 \not\cong T_2$, hence $\mathbb{H}^0(T_1, T_2) = 0$, by Lemma 3.9. As $\mathbb{H}^2(T_2, T_1) = 0$, we have that $\dim \text{Ext}^1(T_2, T_1) = -\chi(T_2, T_1)$. Hence the space parametrizing triples T'' as in (4.5) is the projectivization of a bundle of rank $-\chi(T_2, T_1)$ over U .

Now, for each $(T', T'') \in \mathcal{N}_{\sigma_c}^{\prime, s} \times Y$, we have an exact sequence,

$$\begin{aligned} 0 \rightarrow \text{Hom}(T_2, T') &\rightarrow \text{Hom}(T'', T') \rightarrow \text{Hom}(T_1, T') \rightarrow \\ \text{Ext}^1(T_2, T') &\rightarrow \text{Ext}^1(T'', T') \rightarrow \text{Ext}^1(T_1, T') \rightarrow \\ \mathbb{H}^2(T_2, T') &\rightarrow \mathbb{H}^2(T'', T') \rightarrow \mathbb{H}^2(T_1, T') \rightarrow 0. \end{aligned}$$

As $T_1 \not\cong T'$ and $T_2 \not\cong T'$ (note that $\lambda' > \lambda > \lambda_1, \lambda_2$), we have that $\mathbb{H}^0(T_1, T') = \mathbb{H}^0(T_2, T') = 0$. By assumption $\mathbb{H}^2(T_1, T') = \mathbb{H}^2(T_2, T') = 0$. Therefore $\mathbb{H}^0(T'', T') = \mathbb{H}^2(T'', T') = 0$, and hence $\dim \mathbb{H}^1(T'', T') = -\chi(T'', T')$. The space parametrizing extensions as in (4.4) is a bundle over $\mathcal{N}_{\sigma_c}^{\prime, s} \times Y$ whose fibers are $\text{Ext}^1(T'', T')$. However, not all of them give that T is σ_c^+ -stable. For this it is necessary (and sufficient) that T does not have a subtriple $\tilde{T} \subset T$ with the same σ_c -slope and $\tilde{\lambda} \leq \lambda$. Such \tilde{T} projects to a subtriple of T'' of the same σ_c -slope. But T'' only has one sub-triple of the same σ_c -slope, namely T_1 . If \tilde{T} contains T' , then it is the kernel of $T \twoheadrightarrow T'' \twoheadrightarrow T_2$, and hence $\tilde{\lambda} > \lambda$ (since $\lambda_2 < \lambda$). If not, we must have $\tilde{T} \cong T_1$. Therefore the class of the extension $\xi \in \text{Ext}^1(T'', T')$ defining T lies in the kernel of $\text{Ext}^1(T'', T') \rightarrow \text{Ext}^1(T_1, T')$, hence in the image of

$$\text{Ext}^1(T_2, T') \subset \text{Ext}^1(T'', T').$$

As a conclusion T is σ_c^+ -stable if and only if it is defined by an extension in $A = \text{Ext}^1(T'', T') - \text{Ext}^1(T_2, T')$.

A triple $T \in \mathcal{S}_{\sigma_c^+}$ determines uniquely T' as the only σ_c -stable subtriple with $\lambda' > \lambda$. Hence we must quotient A by the action of $\text{Aut}(T') \times \text{Aut}(T'')$. However, T' is σ_c -stable, hence $\text{Aut}(T') = \mathbb{C}^*$. Also $\text{Aut}(T'') = \mathbb{C}^*$, since it is a non-trivial extension of two non-isomorphic σ_c -stable triples. Therefore the action consists on multiplication by scalars in $\text{Ext}^1(T'', T')$. The result follows. \square

Proposition 4.4. *With the situation as above, let $X^+ \subset \mathcal{S}_{\sigma_c^+}$ (resp. $X^- \subset \mathcal{S}_{\sigma_c^-}$) be the subset of those triples T such that T sits in a non-split exact sequence (4.4), where the exact sequence (4.5) is non-split and $T_1 \cong T_2$. Assume that $\mathbb{H}^2(T_1, T') = \mathbb{H}^2(T_1, T_1) = 0$, for every $T_1 \in \mathcal{N}_{\sigma_c}^{1, s}$, $T' \in \mathcal{N}_{\sigma_c}^{\prime, s}$. Then*

- (1) *Then space Y parametrizing the triples T' (resp. T'') is the projectivization of a bundle of rank $-\chi(T_1, T_1) + 1$ over $\mathcal{N}_{\sigma_c}^{1, s}$.*
- (2) *X^+ (resp. X^-) is a \mathbb{C}^{a-1} -bundle over a \mathbb{P}^{a-1} -bundle over $\mathcal{N}_{\sigma_c}^{\prime, s} \times Y$, where $a = -\chi(T'', T_1)$ (resp. $a = -\chi(T_1, T')$).*

Proof. It is clear that $\mathbb{H}^0(T_1, T_1) = \mathbb{C}$, since T_1 is σ_c -stable. Therefore, $\dim \mathbb{H}^1(T_1, T_1) = -\chi(T_1, T_1) + 1$, and the space Y is a bundle over $\mathcal{N}_{\sigma_c}^{1, s}$ whose fibers are the projective spaces $\mathbb{P}\mathbb{H}^1(T_1, T_1)$.

Fix $(T', T'') \in \mathcal{N}_{\sigma_c}^{\prime, s} \times Y$. Then the space of extensions $\text{Ext}^1(T'', T')$ sits in a short exact sequence

$$0 \rightarrow \text{Ext}^1(T_1, T') \rightarrow \text{Ext}^1(T'', T') \rightarrow \text{Ext}^1(T_1, T') \rightarrow 0. \quad (4.6)$$

As in the proof of Proposition 4.3, T is σ_c^+ -stable if and only if the extension class lies in $A = \text{Ext}^1(T'', T') - \text{Ext}^1(T_1, T')$.

From (4.6), there is a (non canonical) isomorphism $\text{Ext}^1(T'', T') \cong \text{Ext}^1(T_1, T') \oplus \text{Ext}^1(T_1, T')$, so that A gets identified to $\text{Ext}^1(T_1, T') \times (\text{Ext}^1(T_1, T') - \{0\})$. Let us find which extensions give rise to the same (isomorphism class of) T . As T uniquely determines T' and T'' , we must look at the action of $\text{Aut}(T') \times \text{Aut}(T'')$ on $\text{Ext}^1(T'', T')$. Again $\text{Aut}(T') = \mathbb{C}^*$. On the other hand, $0 \rightarrow T_1 \rightarrow T'' \rightarrow T_1 \rightarrow 0$ gives that

$$\text{Aut}(T'') = \mathbb{C} \times \mathbb{C}^*,$$

where $(\lambda, \mu) \in \mathbb{C} \times \mathbb{C}^*$ acts as $(x, y) \mapsto (\mu x + \lambda y, \mu y)$ (for this, choose a \mathcal{C}^∞ splitting $T'' \cong T_1 \oplus T_1$; the splitting is non canonical, but the above formula is independent of the splitting). Therefore, there is an induced action

$$(\lambda, \mu) : (u, v) \mapsto (\mu u + \lambda v, \mu v)$$

on A . The quotient of A by this action is isomorphic to

$$(\mathbb{C}^a \times (\mathbb{C}^a - \{0\})) / \mathbb{C} \times \mathbb{C}^*.$$

Projecting onto the second factor gives a fiber bundle over \mathbb{P}^{a-1} whose fibers are \mathbb{C}^{a-1} . \square

Proposition 4.5. *With the situation as above, let $X^+ \subset \mathcal{S}_{\sigma_c^+}$ (resp. $X^- \subset \mathcal{S}_{\sigma_c^-}$) be the subset of those triples T such that T sits in a non-split exact sequence (4.4), where the exact sequence (4.5) is split and $T_1 \not\cong T_2$ (note that it must be $\lambda_1, \lambda_2 < \lambda$ in this case).*

Assume $\mathbb{H}^2(T_1, T') = \mathbb{H}^2(T_2, T') = 0$, for every T_1, T_2, T' . Then

- (1) *The space Y parametrizing the triples T'' is either $\mathcal{N}_{\sigma_c^+}^{1,s} \times \mathcal{N}_{\sigma_c^+}^{2,s}$, if T_1, T_2 have different types, or $(\mathcal{N}_{\sigma_c^+}^{1,s} \times \mathcal{N}_{\sigma_c^+}^{1,s} - \Delta) / \mathbb{Z}_2$, if T_1, T_2 are of the same type, where Δ is the diagonal, and \mathbb{Z}_2 acts by permutations.*
- (2) *X^+ (resp. X^-) is a bundle over $\mathcal{N}_{\sigma_c^+}^{1,s} \times Y$ (resp. over $Y \times \mathcal{N}_{\sigma_c^+}^{1,s}$) with fibers $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$, where $a = -\chi(T'', T_1)$, $b = -\chi(T'', T_2)$ (resp. $a = -\chi(T_1, T')$, $b = -\chi(T_2, T')$). This fiber bundle is Zariski locally trivial in the first case. In the second case,*

$$X^\pm = (Z \times_S Z - (p \times_S p)^{-1}(\Delta)) / \mathbb{Z}_2,$$

where $p : Z \rightarrow S \times \mathcal{N}_{\sigma_c^+}^{1,s}$, $S = \mathcal{N}_{\sigma_c^+}^{1,s}$ (resp. $S = \mathcal{N}_{\sigma_c^+}^{1,s}$), is a Zariski locally trivial \mathbb{P}^{a-1} -bundle, and \mathbb{Z}_2 acts by permutations.

Proof. The first statement is clear. The extensions parametrizing T are

$$\text{Ext}^1(T'', T') = \text{Ext}^1(T_1, T') \oplus \text{Ext}^1(T_2, T') \cong \mathbb{C}^a \oplus \mathbb{C}^b.$$

By the σ_c -stability, $\mathbb{H}^0(T'', T') = \mathbb{H}^0(T_1, T') \oplus \mathbb{H}^0(T_2, T') = 0$. By assumption, $\mathbb{H}^2(T'', T') = \mathbb{H}^2(T_1, T') \oplus \mathbb{H}^2(T_2, T') = 0$, so $a = \dim \mathbb{H}^1(T_1, T') = -\chi(T_1, T')$ and $b = \dim \mathbb{H}^1(T_2, T') = -\chi(T_2, T')$.

For T to be σ_c^+ -stable, it is necessary and sufficient that it does not contain a subtriple $\tilde{T} \subset T$ of the same σ_c -slope as T , and with $\tilde{\lambda} < \lambda$. Therefore, $\tilde{T} \cong T_1$ or T_2 . If there is an inclusion $T_i = \tilde{T} \hookrightarrow T$, then the extension class defining T lies in the image of

$$\text{Ext}^1(T_j, T') \subset \text{Ext}^1(T'', T')$$

for $j = 3 - i$ ($i = 1, 2$). Therefore the σ_c^+ -stable triples are defined by extensions in

$$\text{Ext}^1(T'', T') - (\text{Ext}^1(T_1, T') \cup \text{Ext}^1(T_2, T')) = (\text{Ext}^1(T_1, T') - \{0\}) \times (\text{Ext}^1(T_2, T') - \{0\}) \quad (4.7)$$

Let us find which extensions give rise to the same (isomorphism class of) T . Any automorphism of T induces automorphisms of T' and T'' . Clearly, $\text{Aut}(T') = \mathbb{C}^*$. On the other hand,

$$\text{Aut}(T'') = \mathbb{C}^* \times \mathbb{C}^*$$

acting diagonally on both factors of $\text{Ext}^1(T'', T') = \text{Ext}^1(T_1, T') \oplus \text{Ext}^1(T_2, T')$ (note that T_1, T_2 are non-isomorphic). If we quotient (4.7) by this action, we get a fiber bundle over Y with fibers

$$(\text{Ext}^1(T_1, T') - \{0\}) \times (\text{Ext}^1(T_2, T') - \{0\}) / \mathbb{C}^* \times \mathbb{C}^* = \mathbb{P} \text{Ext}^1(T_1, T') \times \mathbb{P} \text{Ext}^1(T_2, T').$$

The final assertion is clear, since Z is the fiber bundle with fiber $\mathbb{P} \text{Ext}^1(T_1, T')$ over $(T', T_1) \in S \times \mathcal{N}_{\sigma_c^+}^{1,s}$, which is an algebraic vector bundle. \square

Proposition 4.6. *With the situation as above, let $X^+ \subset \mathcal{S}_{\sigma_c^+}$ (resp. $X^- \subset \mathcal{S}_{\sigma_c^-}$) be the subset of those triples T such that T sits in a non-split exact sequence (4.4), where the exact sequence (4.5) is split and $T_1 \cong T_2$.*

Assume $\mathbb{H}^2(T_1, T') = 0$, for every T_1, T' . Then X^+ (resp. X^-) is a bundle with fibers the grassmannian $\text{Gr}(2, a)$, $a = -\chi(T_1, T')$, over $\mathcal{N}_{\sigma_c^+}^{1,s} \times \mathcal{N}_{\sigma_c^+}^{1,s}$.

Proof. The triples $T'' = T_1 \oplus T_1$ are parametrized by $\mathcal{N}_{\sigma_c}^{1,s}$. Now for any $(T', T_1) \in \mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}^{1,s}$, the extensions (4.4) are parametrized by

$$\mathrm{Ext}^1(T'', T') = \mathrm{Ext}^1(T_1, T') \oplus \mathrm{Ext}^1(T_1, T') = \mathrm{Ext}^1(T_1, T') \otimes \mathbb{C}^2. \quad (4.8)$$

An extension gives rise to a σ_c^+ -unstable triple T if there is a subtriple $\tilde{T} \subset T$ of the same σ_c -slope with $\tilde{\lambda} < \lambda$. The only possibility is that $\tilde{T} \cong T_1$. Composing with the projection $T \rightarrow T''$, we get an embedding $\iota : T_1 \hookrightarrow T'' = T_1 \oplus T_1$. So there exists $(a, b) \in \mathbb{C}^2 - \{0\}$ such that $\iota(x) = (ax, bx)$. The quotient $T''/\iota(\tilde{T})$ is isomorphic to T_1 , and the extension is in the image of

$$\mathrm{Ext}^1(T_1, T') \subset \mathrm{Ext}^1(T_1, T') \otimes \mathbb{C}^2,$$

embedded via $\xi \mapsto (a\xi, b\xi)$. Therefore the σ_c^+ -stable triples correspond to

$$A = \{(\xi_1, \xi_2) \in \mathrm{Ext}^1(T_1, T') \oplus \mathrm{Ext}^1(T_1, T') \mid \xi_1, \xi_2 \text{ linearly independent}\}.$$

To find X^+ , we must quotient by the action of $\mathrm{Aut}(T') = \mathbb{C}^*$ and $\mathrm{Aut}(T'') = \mathrm{GL}(2, \mathbb{C})$ on the space of extensions (4.8). This yields an action of $\mathrm{GL}(2, \mathbb{C})$ on A . The quotient is the grassmannian $\mathrm{Gr}(2, \mathrm{Ext}^1(T_1, T'))$. The result follows. \square

5. HODGE POLYNOMIALS OF THE MODULI SPACES OF TRIPLES OF RANKS $(2, 1)$

In this section we recall the main results of [11]. Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ denote the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ where E_1 is a vector bundle of degree d_1 and rank 2 and E_2 is a line bundle of degree d_2 . By Proposition 3.3, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1/2 - d_2, 2d_1 - 4d_2], \quad \text{where } \mu_1 - \mu_2 \geq 0.$$

Otherwise \mathcal{N}_σ is empty.

Theorem 5.1. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + d_1 - 2d_2$ at the stable points \mathcal{N}_σ^s . Moreover, for non-critical values of σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).* \square

Lemma 5.2 ([11, Lemma 5.3]). *The critical values for the moduli spaces of triples of type $(2, 1, d_1, d_2)$ are the numbers $\sigma_c = 3d_M - d_1 - d_2$ with $\mu_1 \leq d_M \leq d_1 - d_2$. Furthermore, $\sigma_c = \sigma_m \Leftrightarrow d_M = \mu_1$.* \square

The Hodge polynomials of the moduli spaces \mathcal{N}_σ for non-critical values of σ are computed in [11].

Theorem 5.3 ([11, Theorem 6.2]). *Suppose that $\sigma > \sigma_m$ is not a critical value. Set $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil + 1$. Then the Hodge polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ is*

$$e(\mathcal{N}_\sigma) = \mathrm{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+xu)^g(1+xv)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}} \left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x} \right) \right].$$

\square

We also can give the formula for the Hodge polynomial of the moduli space of σ -stable triples when $\sigma > \sigma_m$ is a critical value. Note however that this moduli space is smooth but non-compact, so the Poincaré polynomial is not recovered from the Hodge polynomial.

Proposition 5.4. *Let $\sigma_c = 3\bar{d}_M - d_1 - d_2 > \sigma_m$ be a critical value. Then the Hodge polynomial of the stable part $\mathcal{N}_{\sigma_c}^s$ is*

$$e(\mathcal{N}_{\sigma_c}^s) = \mathrm{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+xu)^g(1+xv)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-\bar{d}_M}} \left(\frac{(uv)^{d_1-d_2-\bar{d}_M}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g+1+2\bar{d}_M}x}{1-(uv)^2x} - 1 \right) \right].$$

Proof. From the Definition 4.1, we easily get that $\mathcal{N}_{\sigma_c}^s = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}$, so

$$e(\mathcal{N}_{\sigma_c}^s) = e(\mathcal{N}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^-}).$$

By part (2) in the proof of [11, Lemma 6.1] (which consists basically on using Proposition 4.2), $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a rank $-\chi(T'', T')$ bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$, where

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 1, d_1 - \bar{d}_M, d_2) = \text{Jac}^{d_2} X \times \text{Sym}^{d_1 - \bar{d}_M - d_2} X, \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 0, \bar{d}_M, 0) = \text{Jac}^{\bar{d}_M} X, \\ -\chi(T'', T') &= 2\bar{d}_M - d_1 + g - 1.\end{aligned}$$

Therefore,

$$\begin{aligned}e(\mathcal{S}_{\sigma_c^-}) &= e(\text{Jac } X)^2 e(\text{Sym}^{d_1 - d_2 - \bar{d}_M} X) e(\mathbb{P}^{2\bar{d}_M - d_1 + g - 2}) \\ &= (1+u)^{2g}(1+v)^{2g} \frac{1 - (uv)^{2\bar{d}_M - d_1 + g - 1}}{1 - uv} \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1 - d_2 - \bar{d}_M}}.\end{aligned}$$

Also $e(\mathcal{N}_{\sigma_c^-})$ is computed by taking $\sigma = \sigma_c^- = \sigma_c - \epsilon$ ($\epsilon > 0$ small) into the formula of Theorem 5.3, with $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil + 1 = \bar{d}_M$. Subtracting both terms, we get

$$\begin{aligned}e(\mathcal{N}_{\sigma_c^-}^s) &= \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+xu)^g(1+xv)^g}{(1-uv)(1-x)(1-uvx)x^{d_1 - d_2 - \bar{d}_M}} \right. \\ &\quad \cdot \left. \left(\frac{(uv)^{d_1 - d_2 - \bar{d}_M}}{1 - (uv)^{-1}x} - \frac{(uv)^{-d_1 + g - 1 + 2\bar{d}_M}}{1 - (uv)^2x} - (1 - (uv)^{2\bar{d}_M - d_1 + g - 1}) \right) \right],\end{aligned}$$

and rearranging we get the stated result. \square

Let $M(2, d)$ denote the moduli space of polystable vector bundles of rank 2 and degree d over X . As $M(2, d) \cong M(2, d + 2k)$, for any integer k , there are two moduli spaces, depending on whether the degree is even or odd. The Hodge polynomial of the moduli space of rank 2 odd degree stable bundles is given in [4, 8, 11]

Theorem 5.5. *The Hodge polynomial of $M(2, d)$ with odd degree d , is*

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}.$$

\square

The Hodge polynomial of the moduli space of rank 2 even degree stable bundles is computed in [12]. Note that this moduli space is smooth but non-compact.

Theorem 5.6 ([12, Theorem A]). *The Hodge polynomial of $M^s(2, d)$ with even degree d , is*

$$\begin{aligned}e(M^s(2, d)) &= \frac{1}{2(1-uv)(1-(uv)^2)} \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g \right. \\ &\quad \left. - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right).\end{aligned}$$

6. CRITICAL VALUES FOR TRIPLES OF RANK $(3, 1)$

Now we move to the analysis of the moduli spaces of σ -polystable triples of rank $(3, 1)$. Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(3, 1, d_1, d_2)$. By Proposition 3.3, σ takes values in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 3(\mu_1 - \mu_2)] = \left[\frac{d_1}{3} - d_2, d_1 - 3d_2 \right].$$

Otherwise \mathcal{N}_σ is empty. Note that if $d_1 - 3d_2 < 0$, then I is empty.

Theorem 6.1. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety of dimension $7g - 6 + d_1 - 3d_2$. It is smooth at any σ -stable point. For non-critical values of σ , we have $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$.*

Proof. Projectiveness follows from Proposition 3.3. Smoothness and the dimension follow from Theorem 3.11. \square

Proposition 6.2. *The critical values σ_c for triples of type $(3, 1, d_1, d_2)$ such that $\sigma_c > \sigma_m$ are the numbers*

$$\sigma_n = 2n - d_1 - d_2, \quad \frac{2}{3}d_1 < n \leq d_1 - d_2.$$

Proof. Let σ_c be a critical value and let $T = (E, L, \phi)$ be a properly σ_c -semistable triple of type $(3, 1, d_1, d_2)$. Let $T' \subset T$ be a σ_c -destabilizing triple. Then we have the following cases:

- (1) If T is σ_c^+ -stable (i.e. $T \in \mathcal{N}_{\sigma_c^+}$) then it must be $\lambda' < \lambda$. Therefore T' must be of type $(n'_1, 0)$, i.e., $T' = (G, 0, 0)$. This gives

$$\frac{d_1 + d_2}{4} + \frac{1}{4}\sigma_c = \mu(G),$$

so $\sigma_c = 4\mu(G) - d_1 - d_2$. If T' is of type $(3, 0)$ then $G = E$ and $\sigma_c = 4\frac{d_1}{3} - d_1 - d_2 = \frac{d_1}{3} - d_2 = \sigma_m$. Otherwise, T' is of type $(2, 0)$ or $(1, 0)$:

- (a) If $T' = (M, 0, 0)$, where M is a line bundle of degree d_M , we have

$$\begin{array}{ccc} 0 & \longrightarrow & M \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & E_1 \\ \downarrow & & \downarrow \\ L & \longrightarrow & F \end{array}$$

and $\sigma_c = 4d_M - d_1 - d_2$. This corresponds to $\sigma_c = \sigma_n$ for $n = 2d_M$.

- (b) If $T' = (F, 0, 0)$, where F is a rank 2 bundle of degree d_F , we have

$$\begin{array}{ccc} 0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & E_1 \\ \downarrow & & \downarrow \\ L & \longrightarrow & L' \end{array}$$

and $\sigma_c = 2d_F - d_1 - d_2$. This corresponds to $\sigma_c = \sigma_n$ for $n = d_F$.

- (2) If T is σ_c^- -stable (i.e. $T \in \mathcal{N}_{\sigma_c^-}$) then it must be $\lambda' > \lambda$. Therefore T' must be of type $(n'_1, 1)$. If T' is of type $(0, 1)$ then it should be $\phi = 0$, which is not possible. So T' is of type $(2, 1)$ or $(1, 1)$. The quotient triple is of the form $T'' = (G, 0, 0)$, with G of rank 1 or 2. So

$$\frac{d_1 + d_2}{4} + \frac{1}{4}\sigma_c = \mu(G)$$

and $\sigma_c = 4\mu(G) - d_1 - d_2$.

- (a) If $T'' = (M, 0, 0)$, where M is a line bundle of degree d_M , we have

$$\begin{array}{ccc} L & \longrightarrow & F \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & E_1 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M \end{array}$$

and $\sigma_c = 4d_M - d_1 - d_2$. This corresponds to $\sigma_c = \sigma_n$ for $n = 2d_M$.

(b) If $T'' = (F, 0, 0)$, where F is a rank 2 bundle of degree d_F , we have

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & E_1 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F \end{array}$$

and $\sigma_c = 2d_F - d_1 - d_2$. This corresponds to $\sigma_c = \sigma_n$ for $n = d_F$.

Finally the condition $\sigma_m < \sigma_c \leq \sigma_M$, i.e. $\frac{d_1}{3} - d_2 < 2n - d_1 - d_2 \leq d_1 - 3d_2$, is translated into $\frac{2}{3}d_1 < n \leq d_1 - d_2$. \square

Let us compute the contribution of a critical value $\sigma_n = 2n - d_1 - d_2 > \sigma_m$ (that is $n > \frac{2}{3}d_1$) to the Hodge polynomial

$$C_n := e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}). \quad (6.1)$$

Let us introduce the following notation

$$\begin{aligned} N_1 &= d_1 - d_2 - n, \\ N_2 &= g - 1 - d_1 + 3\frac{n}{2}. \end{aligned}$$

Proposition 6.3. *If n is odd then C_n in (6.1) equals*

$$\begin{aligned} C_n &= (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\ &\quad \cdot ((uv)^{2N_2} - (uv)^{2N_1}) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}. \end{aligned}$$

Proof. If n is odd, then the only possible cases for properly σ_c -semistable triples are those in cases (1)(b) and (2)(b) of Proposition 6.2 with $d_F = n$ odd. Then F is a semistable bundle of odd degree, hence stable. So $\mathcal{S}_{\sigma_c^+}$ consists of extensions as in (1)(b). Applying Proposition 4.2 (which is possible since $\mathbb{H}^2(T'', T') = 0$ by Lemma 3.10), $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a bundle of rank

$$-\chi(T'', T') = 2d_1 - 2d_F - 2d_2 = 2N_1$$

over

$$\mathcal{N}'_{\sigma_c^s} \times \mathcal{N}''_{\sigma_c^s} = \mathcal{N}_{\sigma_c^s}^s(2, 0, d_F, 0) \times \mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2).$$

Hence

$$e(\mathcal{S}_{\sigma_c^+}) = e(\mathbb{P}^{2N_1-1}) e(\mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2)) e(M(2, d_F)). \quad (6.2)$$

Analogously, $\mathcal{S}_{\sigma_c^-}$ consists of extensions as in (2)(b). Hence, by Proposition 4.2, $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a bundle of rank

$$-\chi(T'', T') = 2g - 2 + 3d_F - 2d_1 = 2N_2$$

over

$$\mathcal{N}'_{\sigma_c^s} \times \mathcal{N}''_{\sigma_c^s} = \mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2) \times \mathcal{N}_{\sigma_c^s}^s(2, 0, d_F, 0).$$

Therefore

$$e(\mathcal{S}_{\sigma_c^-}) = e(\mathbb{P}^{2N_2-1}) e(\mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2)) e(M(2, d_F)). \quad (6.3)$$

Now recall that $d_F = n$ and note that $\mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2) \cong \operatorname{Jac}^{d_2} X \times \operatorname{Sym}^{d_1-d_2-n} X$ (by [11, Lemma 3.16], since $\sigma_c \neq \sigma_m(1, 1, d_1 - d_F, d_2)$). Subtracting (6.2) from (6.3) we get

$$\begin{aligned} C_n &= (e(\mathbb{P}^{2N_1-1}) - e(\mathbb{P}^{2N_2-1})) e(\mathcal{N}_{\sigma_c^s}^s(1, 1, d_1 - d_F, d_2)) e(M(2, d_F)) \\ &= \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-uv} (1+u)^g(1+v)^g \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-n-d_2}} \\ &\quad \cdot \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}. \end{aligned}$$

□

Proposition 6.4. *If n is even then C_n in (6.1) equals*

$$C_n = (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot \left[\left((uv)^{2N_2} - (uv)^{2N_1} \right) \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right. \\ \left. - \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2N_1+1}(1+(uv)^{-2}x)}{1-(uv)^{-1}x} + \frac{(uv)^{2N_2}(1+(uv)^3x)}{1-(uv)^2x} \right. \right. \\ \left. \left. - \frac{(uv)^{N_1+N_2}(1+uv)(1-(uv)x^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)} \right) \right].$$

Proof. Let $\sigma_c = \sigma_n = 2n - d_1 - d_2$. Any triple $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $T \in \mathcal{S}_{\sigma_c^-}$) sits in a non-split exact sequence $T' \rightarrow T \rightarrow T''$ such that T' is σ_c -semistable, T'' is σ_c -stable and $\lambda' < \lambda$ (resp. T' is σ_c -stable, T'' is σ_c -semistable and $\lambda'' < \lambda$). Since $\lambda = \frac{1}{4}$, it must be $\lambda' = 0$ (resp. $\lambda'' = 0$). Therefore, we can decompose $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) into 6 disjoint algebraic locally closed subspaces $\mathcal{S}_{\sigma_c^+} = X_1^+ \cup X_2^+ \cup X_3^+ \cup X_4^+ \cup X_5^+ \cup X_6^+$ (resp. $\mathcal{S}_{\sigma_c^-} = X_1^- \cup X_2^- \cup X_3^- \cup X_4^- \cup X_5^- \cup X_6^-$), as follows

- X_1^+ (resp. X_1^-) consists of those extensions of type (1)(a) (resp. (2)(a)) for which $T'' = (F, L, \phi'')$ is a σ_c -stable triple (resp. $T' = (F, L, \phi')$ is a σ_c -stable triple).
- X_2^+ (resp. X_2^-) consists of those extensions of type (1)(b) (resp. (2)(b)) for which F is a stable bundle of degree $d_F = n$.
- X_3^+ (resp. X_3^-) consists of those extensions of type (1)(b) (resp. (2)(b)) for which F is a properly semistable bundle of degree $d_F = n$, sitting in a non-split exact sequence $L_1 \rightarrow F \rightarrow L_2$, where $L_1 \not\cong L_2$, $L_1, L_2 \in \operatorname{Jac}^{n/2} X$.
- X_4^+ (resp. X_4^-) consists of those extensions of type (1)(b) (resp. (2)(b)) for which F is a properly semistable bundle of degree $d_F = n$, sitting in a non-split exact sequence $L_1 \rightarrow F \rightarrow L_1$, where $L_1 \in \operatorname{Jac}^{n/2} X$.
- X_5^+ (resp. X_5^-) consists of those extensions of type (1)(b) (resp. (2)(b)) for which F is a properly semistable bundle of degree $d_F = n$ of the form $F = L_1 \oplus L_2$, where $L_1 \not\cong L_2$, $L_1, L_2 \in \operatorname{Jac}^{n/2} X$.
- X_6^+ (resp. X_6^-) consists of those extensions of type (1)(b) (resp. (2)(b)) for which F is a properly semistable bundle of degree $d_F = n$ of the form $F = L_1 \oplus L_1$, where $L_1 \in \operatorname{Jac}^{n/2} X$.

We aim to compute

$$\begin{aligned} C_n &= e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}) \\ &= e(\bigsqcup X_i^+) - e(\bigsqcup X_i^-) \\ &= \sum e(X_i^+) - \sum e(X_i^-) \\ &= \sum (e(X_i^+) - e(X_i^-)). \end{aligned}$$

We shall do this by computing each of the terms in the sum above independently:

- (1) We compute $e(X_1^+) - e(X_1^-)$ as follows. Proposition 4.2 implies that X_1^+ is the projectivization of a bundle of rank (with T' and T'' as in (1)(a))

$$-\chi(T'', T') = g - 1 + d_1 - 2d_M - d_2 = g - 1 + d_1 - d_2 - n = g - 1 + N_1.$$

over

$$\mathcal{N}_{\sigma_c}^{\prime, s} \times \mathcal{N}_{\sigma_c}^{\prime\prime, s} = \mathcal{N}_{\sigma_c}^s(1, 0, d_M, 0) \times \mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2).$$

Note that here $d_M = n/2$, and the hypothesis of Proposition 4.2 is satisfied because of Lemma 3.10. Therefore

$$e(X_1^+) = e(\mathbb{P}^{g-1+N_1-1}) e(\mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2)) e(\operatorname{Jac}^{d_M} X). \quad (6.4)$$

Analogously, X_1^- is the projectivization of a bundle of rank (now T' and T'' as in (2)(a))

$$-\chi(T'', T') = 2g - 2 + 3d_M - d_1 = 2g - 2 - d_1 + 3n/2 = g - 1 + N_2$$

over

$$\mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}'' = \mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2) \times \mathcal{N}_{\sigma_c}^s(1, 0, d_M, 0).$$

Therefore

$$e(X_1^-) = e(\mathbb{P}^{g-1+N_2-1}) e(\mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2)) e(\text{Jac}^{d_M} X). \quad (6.5)$$

Proposition 5.4 says that (using $\bar{d}_M = \frac{1}{3}(\sigma_c + d_1 - d_M + d_2) = \frac{1}{3}(2n - d_1 - d_2 + d_1 - \frac{n}{2} + d_2) = \frac{n}{2}$ and $d_M = \frac{n}{2}$),

$$e(\mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2)) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+xu)^g(1+xv)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot \left(\frac{(uv)^{d_1-d_2-n}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g+1+3n/2}x}{1-(uv)^2x} - 1 \right) \right]. \quad (6.6)$$

Note that $\sigma_c > \sigma_m(2, 1, d_1 - d_M, d_2) = (d_1 - d_M)/2 - d_2$ (which follows from $n > \frac{2}{3}d_1$), so Proposition 5.4 applies.

Subtracting (6.5) from (6.4) and using (6.6), we get

$$\begin{aligned} e(X_1^+) - e(X_1^-) &= (e(\mathbb{P}^{g-1+N_1-1}) - e(\mathbb{P}^{g-1+N_2-1})) e(\mathcal{N}_{\sigma_c}^s(2, 1, d_1 - d_M, d_2)) e(\text{Jac}^{d_M} X) \\ &= \frac{(uv)^{g-1}((uv)^{N_2} - (uv)^{N_1})}{1-uv} (1+u)^g(1+v)^g \\ &\quad \cdot \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+xu)^g(1+xv)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-n}} \left(\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right) \right]. \end{aligned}$$

- (2) We compute now $e(X_2^+) - e(X_2^-)$. Any $T \in X_2^+$ is a non-split extension $T' \rightarrow T \rightarrow T''$, where both T' and T'' are σ_c -stable. Moreover, $\mathbb{H}^2(T'', T') = 0$ by Lemma 3.10. So we use Proposition 4.2 to get that X_2^+ is the projectivization of a fiber bundle of rank (with T' and T'' as in (1)(b))

$$-\chi(T'', T') = 2d_1 - 2d_F - 2d_2 = 2N_1$$

over

$$\mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}'' = M^s(2, d_F) \times \mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2).$$

Note that $d_F = n$. Therefore,

$$e(X_2^+) = e(\mathbb{P}^{2N_1-1}) e(\mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2)) e(M^s(2, d_F)). \quad (6.7)$$

Analogously, X_2^- is the projectivization of a fiber bundle of rank (now T' and T'' as in (2)(b))

$$-\chi(T'', T') = 2g - 2 + 3d_F - 2d_1 = 2N_2$$

over

$$\mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}'' = \mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2) \times M^s(2, d_F).$$

Therefore,

$$e(X_2^-) = e(\mathbb{P}^{2N_2-1}) e(\mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2)) e(M^s(2, d_F)). \quad (6.8)$$

Now note that $\mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2) \cong \text{Jac}^{d_2} X \times \text{Sym}^{d_1-d_F-d_2} X$ (using [11, Lemma 3.16], since $\sigma_c \neq \sigma_m(1, 1, d_1 - d_F, d_2)$). Subtracting (6.8) from (6.7) we get

$$\begin{aligned} e(X_2^+) - e(X_2^-) &= (e(\mathbb{P}^{2N_1-1}) - e(\mathbb{P}^{2N_2-1})) e(\mathcal{N}_{\sigma_c}^s(1, 1, d_1 - d_F, d_2)) e(M^s(2, d_F)) \\ &= \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-uv} (1+u)^g(1+v)^g \cdot \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\ &\quad \cdot \frac{1}{2(1-uv)(1-(uv)^2)} \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g \right. \\ &\quad \left. - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right). \end{aligned}$$

- (3) Now we compute $e(X_3^+) - e(X_3^-)$. An element $T \in X_3^+$ sits in a non-split extension $T' \rightarrow T \rightarrow T''$, where $T_1 \rightarrow T'' \rightarrow T_2$ is non-split, and all T', T_1, T_2 are σ_c -stable of the same σ_c -slope, and $T_1 \not\cong T_2$. Note that $T_i = (L_i, 0, 0)$, $i = 1, 2$, since we are in the situation of (1)(b). So $\lambda_1 = 0$, $\lambda_2 = 0$. By Lemma 3.10, we have that $\mathbb{H}^2(T_2, T_1) = \mathbb{H}^2(T_2, T') = \mathbb{H}^2(T_1, T') = 0$. We apply Proposition 4.3, obtaining that X_3^+ is a bundle over $\mathcal{N}'_{\sigma_c} \times Y$, where Y parametrizes the triples T'' , and with fiber $\mathbb{P}^{a-1} - \mathbb{P}^{b-1}$, where

$$\begin{aligned} a &= -\chi(T'', T') = 2d_1 - 4d_M - 2d_2 = 2N_1, \\ b &= -\chi(T_2, T') = d_1 - 2d_M - d_2 = N_1. \end{aligned}$$

The space Y parametrizing non-split extensions $T_1 \rightarrow T'' \rightarrow T_2$ is a projective bundle with fiber \mathbb{P}^{g-2} , since

$$-\chi(T_2, T_1) = g - 1,$$

over

$$\{(T_1, T_2) \in \mathcal{N}_{\sigma_c}(1, 0, n/2, 0) \times \mathcal{N}_{\sigma_c}(1, 0, n/2, 0) \mid T_1 \not\cong T_2\} = (\text{Jac}^{n/2} X \times \text{Jac}^{n/2} X) - \Delta,$$

where Δ stands for the diagonal. Therefore

$$e(X_3^+) = (e(\mathbb{P}^{2N_1-1}) - e(\mathbb{P}^{N_1-1}))e(\text{Jac}^{d_2} X)e(\text{Sym}^{d_1-2d_M-d_2})e((\text{Jac} X \times \text{Jac} X) - \Delta)e(\mathbb{P}^{g-2}). \quad (6.9)$$

The case of X_3^- is analogous. An element $T \in X_3^-$ sits in a non-split extension $T' \rightarrow T \rightarrow T''$, where $T_1 \rightarrow T' \rightarrow T_2$ is non-split, and all T'', T_1, T_2 are σ_c -stable of the same σ_c -slope, and $T_1 \not\cong T_2$. Again $T_i = (L_i, 0, 0)$, $i = 1, 2$. Proposition 4.3 yields that X_3^- is a bundle over

$$\mathcal{N}''_{\sigma_c} \times Y = \text{Jac}^{d_2} X \times \text{Sym}^{d_1-2d_M-d_2} X \times Y,$$

where Y parametrizes the triples T' (therefore Y is a \mathbb{P}^{g-2} -bundle over $(\text{Jac}^{n/2} X \times \text{Jac}^{n/2} X) - \Delta$), and with fiber $\mathbb{P}^{a-1} - \mathbb{P}^{b-1}$, where

$$\begin{aligned} a &= -\chi(T'', T') = 2g - 2 + 6d_M - 2d_1 = 2N_2, \\ b &= -\chi(T'', T_1) = g - 1 + 3d_M - d_1 = N_2. \end{aligned}$$

Therefore

$$e(X_3^-) = (e(\mathbb{P}^{2N_2-1}) - e(\mathbb{P}^{N_2-1}))e(\text{Jac}^{n/2} X)e(\text{Sym}^{d_1-2d_M-d_2})e((\text{Jac} X \times \text{Jac} X) - \Delta)e(\mathbb{P}^{g-2}). \quad (6.10)$$

Subtracting (6.10) from (6.9), we get

$$\begin{aligned} e(X_3^+) - e(X_3^-) &= (e(\mathbb{P}^{2N_1-1}) - e(\mathbb{P}^{N_1-1}) - e(\mathbb{P}^{2N_2-1}) + e(\mathbb{P}^{N_2-1})) \\ &\quad \cdot e(\text{Jac}^{d_2} X)e(\text{Sym}^{d_1-2d_M-d_2})e((\text{Jac} X \times \text{Jac} X) - \Delta)e(\mathbb{P}^{g-2}) \\ &= \frac{(uv)^{2N_2} - (uv)^{N_2} - (uv)^{2N_1} + (uv)^{N_1}}{1 - uv} (1 + u)^g (1 + v)^g \\ &\quad \cdot \text{coeff}_{x^0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - x)(1 - uvx)x^{d_1-d_2-n}} \cdot ((1 + u)^{2g} (1 + v)^{2g} - (1 + u)^g (1 + v)^g) \frac{1 - (uv)^{g-1}}{1 - uv}. \end{aligned}$$

- (4) We move on to compute $e(X_4^+) - e(X_4^-)$. An element $T \in X_4^+$ sits in a non-split extension $T' \rightarrow T \rightarrow T''$, where $T_1 \rightarrow T'' \rightarrow T_1$ is non-split, and T', T_1 are σ_c -stable triples of the same σ_c -slope. The triples $T_1 = (L_1, 0, 0)$ are parametrized by $\text{Jac}^n X$. The triples T'' are then parametrized by a variety Y which is a projective bundle over $\text{Jac}^{n/2} X$ with fiber projective spaces \mathbb{P}^{g-1} , since $\chi(T_1, T_1) = g$. By Lemma 3.10, we have that $\mathbb{H}^2(T_1, T_1) = \mathbb{H}^2(T_1, T') = 0$. By Proposition 4.4, X_4^+ is a bundle over

$$\mathcal{N}'_{\sigma_c} \times Y = \text{Jac}^{d_2} X \times \text{Sym}^{d_1-2d_M-d_2} X \times Y,$$

with fiber a \mathbb{C}^{N_1-1} -bundle over \mathbb{P}^{N_1-1} , as

$$-\chi(T'', T_1) = d_1 - 2d_M - d_2 = N_1.$$

Therefore

$$e(X_4^+) = e(\mathbb{C}^{N_1-1})e(\mathbb{P}^{N_1-1})e(\text{Jac}^{d_2} X)e(\text{Sym}^{d_1-2d_M-d_2})e(\text{Jac}^{n/2} X)e(\mathbb{P}^{g-1}). \quad (6.11)$$

Analogously, an element $T \in X_4^-$ sits in a non-split extension $T' \rightarrow T \rightarrow T''$, where $T_1 \rightarrow T' \rightarrow T_1$ is non-split, and T'', T_1 are σ_c -stable triples of the same σ_c -slope. The triples

T' are parametrized by the variety Y as above. Proposition 4.4 implies that X_4^- is a bundle over

$$Y \times \mathcal{N}_{\sigma_c}^{\prime\prime, s} = Y \times \text{Jac}^{d_2} X \times \text{Sym}^{d_1-2d_M-d_2} X,$$

with fiber a \mathbb{C}^{N_2-1} -bundle over \mathbb{P}^{N_2-1} , as

$$-\chi(T'', T_2) = g - 1 + 3d_M - d_1 = N_2.$$

Hence

$$e(X_4^-) = e(\mathbb{C}^{N_2-1})e(\mathbb{P}^{N_2-1})e(\text{Jac}^{d_2} X)e(\text{Sym}^{d_1-2d_M-d_2})e(\text{Jac}^{n/2} X)e(\mathbb{P}^{g-1}). \quad (6.12)$$

Combining (6.11) with (6.12) we get

$$\begin{aligned} e(X_4^+) - e(X_4^-) &= (e(\mathbb{C}^{N_1-1})e(\mathbb{P}^{N_1-1}) - e(\mathbb{C}^{N_2-1})e(\mathbb{P}^{N_2-1}))e(\text{Jac}^{d_2} X) \\ &\quad \cdot e(\text{Sym}^{d_1-2d_M-d_2})e(\text{Jac}^{n/2} X)e(\mathbb{P}^{g-1}) \\ &= \frac{(uv)^{2N_2-1} - (uv)^{N_2-1} - (uv)^{2N_1-1} + (uv)^{N_1-1}}{1-uv} (1+u)^g(1+v)^g \\ &\quad \cdot \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot (1+u)^g(1+v)^g \frac{1-(uv)^g}{1-uv}. \end{aligned}$$

- (5) We proceed with $e(X_5^+) - e(X_5^-)$. An element $T \in X_5^+$ sits in a non-split exact sequence $T' \rightarrow T \rightarrow T''$ where $T'' = T_1 \oplus T_2$, with T_1, T_2 non-isomorphic triples, and T', T_1, T_2 σ_c -stable triples of the same σ_c -slope. Here $T_i = (L_i, 0, 0)$ as in (1)(b). So the space parametrizing T'' is

$$Y = (\text{Jac}^{n/2} X \times \text{Jac}^{n/2} X - \Delta)/\mathbb{Z}_2,$$

where \mathbb{Z}_2 acts by permutation $(T_1, T_2) \mapsto (T_2, T_1)$.

By Lemma 3.10, $\mathbb{H}^2(T_1, T') = \mathbb{H}^2(T_2, T') = 0$. Then Proposition 4.5 implies that X_5^+ is a bundle over $S \times Y$, $S = \mathcal{N}_{\sigma_c}^{\prime\prime, s}$, whose fibers are $\mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$, with

$$a = -\chi(T'', T_1) = d_1 - 2d_M - d_2 = N_1.$$

However, this fiber bundle is not locally trivial in the Zariski topology, since it has monodromy around the diagonal. We compute the Hodge polynomial of X_5^+ as follows. Pull-back the bundle to $\tilde{Y} = S \times (\text{Jac}^{n/2} X \times \text{Jac}^{n/2} X - \Delta)$,

$$\begin{array}{ccc} \tilde{X}_5^+ & \longrightarrow & X_5^+ \\ \downarrow & & \downarrow \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

Then \tilde{X}_5^+ is a $\mathbb{P}^{N_1-1} \times \mathbb{P}^{N_1-1}$ -bundle over \tilde{Y} and $X_5^+ = \tilde{X}_5^+/\mathbb{Z}_2$. More explicitly, let Z be the projective bundle over $S \times \text{Jac}^{n/2} X$, with fibers $\text{Ext}^1(T_1, T')$, and let $p : Z \times_S Z \rightarrow S \times \text{Jac}^{n/2} X \times \text{Jac}^{n/2} X$ stand for the projection. Then $\tilde{X}_5^+ = p^{-1}(S \times (\text{Jac}^{n/2} X \times \text{Jac}^{n/2} X - \Delta))$. Letting \mathbb{Z}_2 act on $Z \times_S Z$ by permutation, we have an induced map $p : (Z \times_S Z)/\mathbb{Z}_2 \rightarrow (S \times \text{Jac}^{n/2} X \times \text{Jac}^{n/2} X)/\mathbb{Z}_2$ and $X_5^+ = p^{-1}(Y)$. Now

$$e((Z \times_S Z)/\mathbb{Z}_2) = e(S) \cdot \frac{1}{2} \left(e(\mathbb{P}^{N_1-1})^2 e(\text{Jac} X)^2 + \frac{1-(uv)^{2N_1}}{1-(uv)^2} (1-u^2)^g (1-v^2)^g \right),$$

using (7) of Section 2. Also $p^{-1}(S \times \Delta)$ is a bundle over $S \times \Delta \cong S \times \text{Jac}^{n/2} X$, whose fibers are $(\mathbb{P}^{N_1-1} \times \mathbb{P}^{N_1-1})/\mathbb{Z}_2$, where \mathbb{Z}_2 acts by permutation. Hence

$$e(p^{-1}(S \times \Delta)) = e(S) e(\text{Jac} X) \cdot \frac{1}{2} \left(e(\mathbb{P}^{N_1-1})^2 + \frac{1-(uv)^{2N_1}}{1-(uv)^2} \right),$$

using (7) of Section 2 again. We finally get

$$e(X_5^+) = e((Z \times_S Z)/\mathbb{Z}_2) - e(p^{-1}(S \times \Delta)), \quad (6.13)$$

and $e(S) = e(\text{Jac}^{d_2} X) e(\text{Sym}^{d_1-d_F-d_2} X)$.

There is an analogous formula for $e(X_5^-)$, obtained by substituting N_1 by N_2 in (6.13). So we obtain

$$\begin{aligned}
e(X_5^+) - e(X_5^-) &= (1+u)^g(1+v)^g \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\
&\quad \cdot \frac{1}{2} \left(\frac{(1-(uv)^{N_1})^2}{(1-uv)^2} (1+u)^{2g}(1+v)^{2g} + \frac{1-(uv)^{2N_1}}{1-(uv)^2} (1-u^2)^g(1-v^2)^g \right. \\
&\quad \left. - (1+u)^g(1+v)^g \frac{(1-(uv)^{N_1})^2}{(1-uv)^2} - (1+u)^g(1+v)^g \frac{1-(uv)^{2N_1}}{1-(uv)^2} \right. \\
&\quad \left. - \frac{(1-(uv)^{N_2})^2}{(1-uv)^2} (1+u)^{2g}(1+v)^{2g} - \frac{1-(uv)^{2N_2}}{1-(uv)^2} (1-u^2)^g(1-v^2)^g \right. \\
&\quad \left. + (1+u)^g(1+v)^g \frac{(1-(uv)^{N_2})^2}{(1-uv)^2} + (1+u)^g(1+v)^g \frac{1-(uv)^{2N_2}}{1-(uv)^2} \right) \\
&= (1+u)^g(1+v)^g \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\
&\quad \cdot \frac{1}{2} \left(\frac{(uv)^{2N_1} - 2(uv)^{N_1} - (uv)^{2N_2} + 2(uv)^{N_2}}{(1-uv)^2} ((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g) \right. \\
&\quad \left. + \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-(uv)^2} ((1-u^2)^g(1-v^2)^g - (1+u)^g(1+v)^g) \right).
\end{aligned}$$

- (6) We deal with the last case, $e(X_6^+) - e(X_6^-)$. A triple $T \in X_6^+$ sits in a non-split extension $T' \rightarrow T \rightarrow T''$, where $T'' = T_1 \oplus T_1$ and T', T_1 are σ_c -stable triples with the same σ_c -slope. Here $T_1 = (L_1, 0, 0)$ as in (1)(b). Using that $\mathbb{H}^1(T_1, T') = 0$ (by Lemma 3.10), Proposition 4.6 implies that X_6^+ is a grassmannian bundle over

$$\mathcal{N}'_{\sigma_c} \times \operatorname{Jac}^{n/2} X = \operatorname{Jac}^{d_2} X \times \operatorname{Sym}^{d_1-d_F-d_2} X \times \operatorname{Jac}^n X,$$

with fibers $\operatorname{Gr}(2, N_1)$, since

$$-\chi(T_1, T') = N_1.$$

Hence

$$e(X_6^+) = e(\operatorname{Jac}^{d_2} X) e(\operatorname{Sym}^{d_1-d_F-d_2} X) e(\operatorname{Jac}^{d_M} X) e(\operatorname{Gr}(2, N_1)).$$

Analogously,

$$e(X_6^-) = e(\operatorname{Jac}^{d_2} X) e(\operatorname{Sym}^{d_1-d_F-d_2} X) e(\operatorname{Jac}^{d_M} X) e(\operatorname{Gr}(2, N_2)),$$

and then

$$\begin{aligned}
e(X_6^+) - e(X_6^-) &= e(\operatorname{Jac}^{d_2} X) e(\operatorname{Sym}^{d_1-d_F-d_2} X) e(\operatorname{Jac}^{d_M} X) (e(\operatorname{Gr}(2, N_1)) - e(\operatorname{Gr}(2, N_2))) \\
&= (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\
&\quad \cdot \frac{(1-(uv)^{N_1})(1-(uv)^{N_1-1}) - (1-(uv)^{N_2})(1-(uv)^{N_2-1})}{(1-uv)(1-(uv)^2)} \\
&= (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_F-d_2}} \\
&\quad \cdot \frac{(uv)^{N_2} + (uv)^{N_2-1} - (uv)^{2N_2-1} - (uv)^{N_1} - (uv)^{N_1-1} + (uv)^{2N_1-1}}{(1-uv)(1-(uv)^2)}.
\end{aligned}$$

Putting all together,

$$\begin{aligned}
C_n = & (1+u)^g(1+v)^g \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-n-d_2}} \cdot \\
& \cdot \left[\frac{(uv)^{N_2} - (uv)^{N_1}}{1-uv} (uv)^{g-1} \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)} \left(\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right) \right. \\
& + \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-uv} \cdot \frac{1}{2(1-uv)(1-(uv)^2)} \cdot \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g \right. \\
& \quad \left. - (1+u)^{2g}(1+v)^{2g}(1+2(uv)^{g+1} - (uv)^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right) \\
& + \frac{(uv)^{2N_2} - (uv)^{N_2} - (uv)^{2N_1} + (uv)^{N_1}}{1-uv} ((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g) \frac{1-(uv)^{g-1}}{1-uv} \\
& + \frac{(uv)^{2N_2-1} - (uv)^{N_2-1} - (uv)^{2N_1-1} + (uv)^{N_1-1}}{1-uv} (1+u)^g(1+v)^g \frac{1-(uv)^g}{1-uv} \\
& + \frac{1}{2} \left(\frac{(uv)^{2N_1} - 2(uv)^{N_1} - (uv)^{2N_2} + 2(uv)^{N_2}}{(1-uv)^2} ((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g) \right. \\
& \quad \left. + \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-(uv)^2} ((1-u^2)^g(1-v^2)^g - (1+u)^g(1+v)^g) \right) \\
& \left. + (1+u)^g(1+v)^g \frac{(uv)^{N_2} + (uv)^{N_2-1} - (uv)^{2N_2-1} - (uv)^{N_1} - (uv)^{N_1-1} + (uv)^{2N_1-1}}{(1-uv)(1-(uv)^2)} \right],
\end{aligned}$$

Rearranging, we get

$$\begin{aligned}
C_n = & (1+u)^g(1+v)^g \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot \\
& \cdot \left[\frac{(uv)^{N_2} - (uv)^{N_1}}{1-uv} (uv)^{g-1} \frac{(1+u)^{2g}(1+v)^{2g}}{1-uv} \left(\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right) \right. \\
& + ((uv)^{2N_2} - (uv)^{2N_1}) \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^{g-1}(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)} \\
& \left. + ((uv)^{N_2} - (uv)^{N_1})(1+u)^{2g}(1+v)^{2g} \frac{(uv)^{g-1}}{(1-uv)^2} \right] \\
= & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot \\
& \cdot \left[\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \left(\frac{(uv)^{N_1+N_2} - (uv)^{2N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{2N_2+2}x - (uv)^{N_1+N_2+2}x}{1-(uv)^2x} \right) \right. \\
& \left. + ((uv)^{2N_2} - (uv)^{2N_1}) \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right] \\
= & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \cdot \\
& \cdot \left[\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \left(-\frac{(uv)^{2N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{2N_2+2}x}{1-(uv)^2x} - \frac{(uv)^{2N_2} - (uv)^{2N_1}}{1-(uv)^2} (1-uv) \right) \right. \\
& + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \left(\frac{(uv)^{N_1+N_2}}{1-(uv)^{-1}x} + \frac{(uv)^{N_1+N_2+2}x}{1-(uv)^2x} \right) \\
& \left. + ((uv)^{2N_2} - (uv)^{2N_1}) \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right],
\end{aligned}$$

hence

$$\begin{aligned}
C_n = & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2-n}} \\
& \left[-\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2N_1+1}(1+(uv)^{-2}x)}{1-(uv)^{-1}x} + \frac{(uv)^{2N_2}(1+(uv)^3x)}{1-(uv)^2x} \right) \right. \\
& + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} (uv)^{N_1+N_2} \frac{1-(uv)x^2}{(1-(uv)^{-1}x)(1-(uv)^2x)} \\
& \left. + ((uv)^{2N_2} - (uv)^{2N_1}) \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right].
\end{aligned}$$

The result follows. \square

Theorem 6.5. *Let $\sigma > \sigma_m$ be a non-critical value. Set*

$$n_0 = \left\lfloor \frac{\sigma + d_1 + d_2}{2} \right\rfloor \quad \text{and} \quad \bar{n}_0 = 2 \left\lfloor \frac{n_0 + 1}{2} \right\rfloor.$$

Then the Hodge polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(3, 1, d_1, d_2)$ is

$$\begin{aligned}
e(\mathcal{N}_\sigma) = & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} \\
& \cdot \left[\left(\frac{(uv)^{2d_1-2d_2-2n_0}x^{n_0}}{1-(uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0}x^{n_0}}{1-(uv)^3x} \right) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right. \\
& + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1}x^{\bar{n}_0}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} \right. \\
& \left. \left. + \frac{(uv)^{2g-2-2d_1+3\bar{n}_0}x^{\bar{n}_0}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1-d_2+\bar{n}_0/2}x^{\bar{n}_0}}{(1-(uv)^2x)(1-(uv)^{-1}x)} \right) \right].
\end{aligned}$$

Proof. We have the following telescopic sum

$$e(\mathcal{N}_\sigma) = \sum_{\sigma_c > \sigma} (e(\mathcal{N}_{\sigma_c^-}) - e(\mathcal{N}_{\sigma_c^+})) = \sum_{n \geq n_0} (-C_n),$$

since $\sigma_c = 2n - d_1 - d_2 > \sigma$ is equivalent to $n > \frac{\sigma+d_1+d_2}{2}$, i.e. $n \geq n_0$. Note, incidentally, that σ is non-critical is equivalent to $\frac{\sigma+d_1+d_2}{2}$ not being an integer. Using Propositions 6.3 and 6.4, we get

$$\begin{aligned}
e(\mathcal{N}_\sigma) = & \sum_{n \geq n_0} (-C_n) = \sum_{n \geq n_0, n \text{ odd}} (-C_n) + \sum_{n \geq n_0, n \text{ even}} (-C_n) \\
= & (1+u)^{2g}(1+v)^{2g} \operatorname{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} \\
& \left[\sum_{n \geq n_0} x^n ((uv)^{2N_1} - (uv)^{2N_2}) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} + \right. \\
& + \sum_{n \geq n_0, n \text{ even}} x^n \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2N_1+1}(1+(uv)^{-2}x)}{1-(uv)^{-1}x} + \frac{(uv)^{2N_2}(1+(uv)^3x)}{1-(uv)^2x} \right. \\
& \left. \left. - \frac{(uv)^{N_1+N_2}(1+uv)(1-(uv)x^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)} \right) \right].
\end{aligned}$$

Recall that $N_1 = d_1 - d_2 - n$, $N_2 = g - 1 + 3n/2 - d_1$. Also, note that \bar{n}_0 is the first even number greater than or equal to n_0 . We substitute

$$\begin{aligned} \sum_{n \geq n_0} x^n ((uv)^{2N_1} - (uv)^{2N_2}) &= \sum \left(x^n (uv)^{2d_1-2d_2-2n} - x^n (uv)^{2g-2-2d_1+3n} \right) \\ &= \frac{(uv)^{2d_1-2d_2-2n_0} x^{n_0}}{1 - (uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0} x^{n_0}}{1 - (uv)^3x}, \\ \sum_{n \geq n_0, n \text{ even}} x^n (uv)^{2N_1+1} &= \sum x^n (uv)^{2d_1-2d_2-2n+1} = \frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1} x^{\bar{n}_0}}{1 - (uv)^{-4}x^2}, \\ \sum_{n \geq n_0, n \text{ even}} x^n (uv)^{2N_2} &= \sum x^n (uv)^{2g-2-2d_1+3n} = \frac{(uv)^{2g-2-2d_1+3\bar{n}_0} x^{\bar{n}_0}}{1 - (uv)^6x^2}, \\ \sum_{n \geq n_0, n \text{ even}} x^n (uv)^{N_1+N_2} &= \sum x^n (uv)^{g-1-d_2+n/2} = \frac{(uv)^{g-1-d_2+\bar{n}_0/2} x^{\bar{n}_0}}{1 - uvx^2}, \end{aligned}$$

in the formula above, to get

$$\begin{aligned} e(\mathcal{N}_\sigma) &= (1+u)^{2g}(1+v)^{2g} \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} \\ &\quad \left[\left(\frac{(uv)^{2d_1-2d_2-2n_0} x^{n_0}}{1 - (uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0} x^{n_0}}{1 - (uv)^3x} \right) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right. \\ &\quad + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1} x^{\bar{n}_0} (1+(uv)^{-2}x)}{(1-(uv)^{-4}x^2)(1-(uv)^{-1}x)} \right. \\ &\quad \left. \left. + \frac{(uv)^{2g-2-2d_1+3\bar{n}_0} x^{\bar{n}_0} (1+(uv)^3x)}{(1-(uv)^6x^2)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1-d_2+\bar{n}_0/2} x^{\bar{n}_0}}{(1-(uv)^2x)(1-(uv)^{-1}x)} \right) \right]. \end{aligned}$$

Simplifying,

$$\begin{aligned} e(\mathcal{N}_\sigma) &= \text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} (1+u)^{2g}(1+v)^{2g} \\ &\quad \left[\left(\frac{(uv)^{2d_1-2d_2-2n_0} x^{n_0}}{1 - (uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0} x^{n_0}}{1 - (uv)^3x} \right) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right. \\ &\quad + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1} x^{\bar{n}_0}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} \right. \\ &\quad \left. \left. + \frac{(uv)^{2g-2-2d_1+3\bar{n}_0} x^{\bar{n}_0}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1-d_2+\bar{n}_0/2} x^{\bar{n}_0}}{(1-(uv)^2x)(1-(uv)^{-1}x)} \right) \right], \end{aligned}$$

as required. \square

7. HODGE POLYNOMIAL OF THE MODULI SPACE OF RANK 3 STABLE BUNDLES

Now we want to use Proposition 3.4 to compute the Hodge polynomial for the moduli space $M(3, 1)$. Note that $M(3, 1) \cong M(3, -1)$, via $E \mapsto E^*$. Also $M(3, d) \cong M(3, d+3k)$, for any $k \in \mathbb{Z}$, by twisting with a fixed line bundle of degree k . Therefore all $M(3, d)$, with $d \not\equiv 0 \pmod{3}$ are isomorphic to each other.

Theorem 7.1. *Assume that $d \not\equiv 0 \pmod{3}$. Then the Hodge polynomial of $M(3, d)$ is*

$$\begin{aligned} e(M(3, d)) &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)^2(1-(uv)^3)} \left((1+u)^g(1+v)^g(1+uv)^2(uv)^{2g-1}(1+u^2v)^g(1+uv^2)^g \right. \\ &\quad \left. - (1+u)^{2g}(1+v)^{2g}(uv)^{3g-1}(1+uv+u^2v^2) + (1+u^2v^3)^g(1+u^3v^2)^g(1+u^2v)^g(1+uv^2)^g \right). \end{aligned}$$

Proof. We choose $d_2 = 0$, $d_1 = 6g - 5$. By Proposition 3.4,

$$\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(3, 1, d_1, d_2) \rightarrow \text{Jac}^{d_2} X \times M(3, d_1)$$

is a projective bundle with projective fibers of dimension $d_1 - 3(g - 1) - 1 = 3g - 3$. Then

$$e(\mathcal{N}_{\sigma_m^+}) = e(M(3, d_1)) (1 + u)^g (1 + v)^g \frac{1 - (uv)^{3g-2}}{1 - uv}.$$

To compute $e(\mathcal{N}_{\sigma_m^+})$, apply Theorem 6.5, with $\sigma = \sigma_m^+ = \frac{d_1}{3} + \epsilon$ ($\epsilon > 0$ small), so $n_0 = \lceil \frac{\sigma + d_1 + d_2}{2} \rceil = \lceil \frac{2}{3}d_1 \rceil + 1 = 4g - 3$, $\bar{n}_0 = 4g - 2$, and $d_1 - d_2 - n_0 = 2g - 2$, to get

$$\begin{aligned} e(\mathcal{N}_{\sigma_m^+}) = & (1 + u)^{2g} (1 + v)^{2g} \text{coeff}_{x^0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - x)(1 - uvx)x^{2g-2}} \cdot \\ & \cdot \left[\left(\frac{(uv)^{4g-4}}{1 - (uv)^{-2}x} - \frac{(uv)^{2g-1}}{1 - (uv)^3x} \right) \cdot \frac{(1 + u^2v)^g (1 + uv^2)^g - (uv)^g (1 + u)^g (1 + v)^g}{(1 - uv)^2 (1 - (uv)^2)} \right] \\ & + (1 + u)^{2g} (1 + v)^{2g} \text{coeff}_{x^0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - x)(1 - uvx)x^{2g-1}} \cdot \\ & \cdot \left[\frac{(uv)^{g-1} (1 + u)^g (1 + v)^g}{(1 - uv)^2 (1 + uv)} \left(\frac{(uv)^{4g-5}}{(1 - (uv)^{-2}x)(1 - (uv)^{-1}x)} \right. \right. \\ & \left. \left. + \frac{(uv)^{2g+2}}{(1 - (uv)^3x)(1 - (uv)^2x)} - \frac{(1 + uv)(uv)^{3g-2}}{(1 - (uv)^2x)(1 - (uv)^{-1}x)} \right) \right]. \end{aligned}$$

Introducing the notation

$$F_1(a, b, c) = \text{Res}_{x=0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - ax)(1 - bx)(1 - cx)x^{2g-1}},$$

$$F_2(a, b, c, d) = \text{Res}_{x=0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - ax)(1 - bx)(1 - cx)(1 - dx)x^{2g-2}},$$

we write

$$\begin{aligned} e(\mathcal{N}_{\sigma_m^+}) = & (1 + u)^{2g} (1 + v)^{2g} \left[\left((uv)^{4g-4} F_1(1, uv, (uv)^{-2}) - (uv)^{2g-1} F_1(1, uv, (uv)^3) \right) \cdot \right. \\ & \cdot \frac{(1 + u^2v)^g (1 + uv^2)^g - (uv)^g (1 + u)^g (1 + v)^g}{(1 - uv)^2 (1 - (uv)^2)} \\ & + \frac{(uv)^{g-1} (1 + u)^g (1 + v)^g}{(1 - uv)^2 (1 + uv)} \left((uv)^{4g-5} F_2(1, uv, (uv)^{-2}, (uv)^{-1}) \right. \\ & \left. \left. + (uv)^{2g+2} F_2(1, uv, (uv)^3, (uv)^2) - (1 + uv)(uv)^{3g-2} F_2(1, uv, (uv)^2, (uv)^{-1}) \right) \right]. \end{aligned}$$

In the proof of [11, Proposition 8.1]), we computed

$$F_1(a, b, c) = \frac{(a + u)^g (a + v)^g}{(a - b)(a - c)} + \frac{(b + u)^g (b + v)^g}{(b - a)(b - c)} + \frac{(c + u)^g (c + v)^g}{(c - a)(c - b)}.$$

Also, the function

$$G(x) = \frac{(1 + ux)^g (1 + vx)^g}{(1 - ax)(1 - bx)(1 - cx)(1 - dx)x^{2g-2}},$$

is meromorphic on $\mathbb{C} \cup \{\infty\}$ with poles at $x = 0$, $x = 1/a$, $x = 1/b$, $x = 1/c$ and no pole at ∞ . So

$$F_2(a, b, c, d) = -\text{Res}_{x=1/a} G(x) - \text{Res}_{x=1/b} G(x) - \text{Res}_{x=1/c} G(x) - \text{Res}_{x=1/d} G(x),$$

from where

$$F_2(a, b, c, d) = \frac{(a + u)^g (a + v)^g}{(a - b)(a - c)(a - d)} + \frac{(b + u)^g (b + v)^g}{(b - a)(b - c)(b - d)} + \frac{(c + u)^g (c + v)^g}{(c - a)(c - b)(c - d)} + \frac{(d + u)^g (d + v)^g}{(d - a)(d - b)(d - d)}.$$

Substituting in the above, and simplifying, we get

$$\begin{aligned} e(M(3, d_1)) &= \frac{e(\mathcal{N}_{\sigma_m^+})(1-uv)}{(1+u)^g(1+v)^g(1-(uv)^{3g-2})} \\ &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)^2(1-(uv)^3)} \left((1+u)^g(1+v)^g(1+uv)^2(uv)^{2g-1}(1+u^2v)^g(1+uv^2)^g \right. \\ &\quad \left. - (1+u)^{2g}(1+v)^{2g}(uv)^{3g-1}(1+uv+u^2v^2) + (1+u^2v^3)^g(1+u^3v^2)^g(1+u^2v)^g(1+uv^2)^g \right). \end{aligned}$$

□

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